Analytical Tableaux for da Costa's Paraconsistent Predicate Calculi C_n*[1]

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ABSTRACT: *In this paper, based on Castro(2004) and D´Ottaviano and Castro (2005), we introduce a hierarchy of syntactical tableaux systems* **TNDC**$_n$*, $1 \leq n < \omega$, *for da Costa's hierarchy of predicate paraconsistent logics* **C**$_n$*, $1 \leq n < \omega$. *In our tableaux formulation for the* **TNDC**$_n$*, *as in our paper of 2005, we introduce da Costa's ´ball´ operator 'o', the generalized operators 'k', '(k)' and the negations '$\sim_k$', for $k \geq 1$, as primitive operators. In this generalization it is necessary to deal with specific problems, concerning relationships between the mentioned generalized distinct operators; and relationships between the different systems of the hierarchy* **TNDC**$_n$*. *Another peculiarity is that we define two conditions for the closure of the branches. We prove a generalized version of the Cut Rule for the* **TNDC**$_n$* *and also prove that these systems are logically equivalent to the corresponding da Costa's calculi. Our systems* **TNDC**$_n$* *are introduced from a denumerable(infinite) set of primitive operators and this allows us to capture* **C**$_n$* *as paraconsistent extensions of classical logic. The systems* **TNDC**$_n$* *constitute completely automated theorem proving systems for the systems of da Costa's hierarchy* **C**$_n$*, $1 \leq n < \omega$. *As far as we know, besides introducing every one of the operators of the three da Costa's families of operators 'k', '(k)' and the negations '$\sim_k$', for $k \geq 1$, as primitive, this is the first paper in the literature in which the hierarchy of logics* **C**$_n$*, $1 \leq n < \omega$, *receive a tableaux approach.*

KEY WORDS: *paraconsistent logic; paraconsistent first-order predicate calculi; da Costa's hierarchy of paraconsistent first-order predicate calculi* **C**$_n$*, $1 \leq n < \omega$; *hierarchy of analytical tableaux systems; cut rule; logical equivalence; decidability*

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Introduction

A theory **T**, whose language has a symbol for negation, is said to be *inconsistent* (*contradictory*) if it has as theorems a formula and its negation; otherwise, **T** is *consistent* (*non-contradictory*). A theory **T** is said to be *trivial* if every formula of its language is a theorem; otherwise, T is *non-trivial*.

A logic is *paraconsistent* if it can be used as the underlying logic to inconsistent but non-trivial theories, which we call *paraconsistent theories*.

In paraconsistent logic the role of the Principle of Non-Contradiction is, in a certain sense, restricted. Although in those logics the Principle of Non-Contradiction is not necessarily invalid, from a formula and its negation it is not possible, in general, to deduce any formula (see, for instance, D'Ottaviano (1990)).

In this paper, we introduce a new hierarchy of predicate analytical tableaux systems, the $\boldsymbol{TNDC_n}^*$, $1 \leq n < \omega$, for da Costa's hierarchy of predicate calculi for the study of inconsistent but non-trivial theories (da Costa 1963a, 1963b, 1964a, 1964b and 1964c, 1974): the hierarchy of first-order predicate calculi $\boldsymbol{C_n}^*$, $1 \leq n < \omega$.

In our hierarchy $\boldsymbol{TNDC_n}^*$, $1 \leq n < \omega$, as in the hierarchy of propositional analytical tableaux systems $\boldsymbol{TNDC_n}$, $1 \leq n < \omega$, introduced by Castro (2004) and D'Ottaviano and Castro (2005), the 'ball' operator 'o' is introduced as primitive, as well as the generalized operators 'k' and '(k)', and the negations '$\sim_k$', for $k \geq 1$, are primitive operators. As in the case of our previous paper, it is necessary to deal with specific problems concerning relationships between the generalized distinct primitive operators; and with relationships between the different systems of the hierarchy $\boldsymbol{TNDC_n}^*$, $1 \leq n < \omega$. We also define two conditions for the closure of the branches of the tableaux, looking for reflecting the meaning of the paraconsistent negation: either they are closed by the strong negation '$\sim_n$', as usual, or they are closed by the paraconsistent negation '$\neg$' and additional conditions.

We prove a general version of the Cut Rule (Theorem) for $\boldsymbol{TNDC}_{\mathrm{n}}^*$, $1 \leq n < \omega$, and also prove that these systems are logically equivalent to the corresponding systems $\boldsymbol{C}_{\mathrm{n}}^*$, $1 \leq n < \omega$, respectively.

As far as we know in the literature, the first paraconsistent system in which da Costa's 'ball' operator 'o' was treated as a primitive "consistency" operator was introduced by D´Ottaviano and Epstein (1988), in a modified version of the system $\mathbf{J}_3$ of D´Ottaviano and da Costa (1970), also presented in Epstein (1990, 1995), Chapter IX, written in collaboration with D'Ottaviano.

De Castro presented, for the first time, our conception concerning the primitiveness of the operators of the families 'k', '(k)' and the negations '$\sim_k$' for the construction of the hierarchy of tableaux systems $\boldsymbol{TNDC}_n$, $1\leq n<\omega$, in his communication during the "II World Congress on Paraconsistency (II WCP)", held in Juquehy, Brazil, in 2000 (see Castro (2000)).

In some more recent papers Carnielli, Coniglio and Marcos have studied an ample class of paraconsistent axiomatic propositional systems (Hilbert style), in which a unary "consistency operator" is introduced as a primitive operator of the object language (see Carnielli and Marcos (2002) and Carnielli, Coniglio and Marcos (2007)). The system $\boldsymbol{J}_3$ is a particular logic of their class of paraconsistent systems; da Costa's propositional systems $\boldsymbol{C}_n$, $1\leq n<\omega$, are seen as a primary sub-class of that class, such that in $\boldsymbol{C}_1$, the primitive "consistency operator" corresponds to da Costa's defined "ball" operator "o".

We remind that Marconi (1980) is the first tableaux system for da Costa's calculus $\boldsymbol{C}_1$, the first logic of the hierarchy of propositional calculi $\boldsymbol{C}_n$, $1\leq n\leq\omega$. He introduces a variant of semantic tableaux systems, *à la* Beth (see Beth (1959)), in order to prove the completeness and decidability of da Costa's propositional system $\boldsymbol{C}_1$. He also claims that his method could be expanded for the systems $\boldsymbol{C}_n$, $2\leq n<\omega$.

Carnielli and Lima-Marques (1992) introduce a semantic tableaux system, *à la* Smullyan (see Smullyan (1968)), for Alves's paraconsistent propositional logic[2] $\boldsymbol{C}_1^{1}$, and for the paraconsistent predicate logic with equality $\boldsymbol{C}_1^{1=}$ (see Alves (1976)), namely the systems $\boldsymbol{TC}_1$ and $\boldsymbol{TC}_1^{=}$ respectively, and show that these systems are complete and decidable.

Buchsbaum and Pequeno (1993) introduce a syntactical tableaux system, also *à la* Smullyan, for da Costa's $\boldsymbol{C}_1$*, the system *S***C**1*, showing that *S***C**1* is complete.

In the system *S***C**1* of Buchsbaum and Pequeno we do not have an explicit rule that determines *a priori* when the definition of the operator 'o' must be used or must not be used during the derivations; on account of this it is possible to occur open branches that must be rebuilt, in a distinct way, from the mentioned occurrence of the operator 'o'.

Also in Carnielli and Lima-Marques's systems $\boldsymbol{TC}_1$ and $\boldsymbol{TC}_1^{=}$ there are not specific rules that determine *a priori* when to use the definition of the operator 'o', what may make necessary to rebuild branches. Particularly, in these systems infinite loops may occur, 'postponing indefinitely', according to the own authors, the analysis of formulae that involve the operator of primitive negation and, as a natural consequence, the operator 'o'; Carnielli and Lima-Marques prove the decidability of $\boldsymbol{TC}_1$ and $\boldsymbol{TC}_1^{=}$, showing how to deal with the infinite loops. Carnielli, Coniglio and Marcos (2007) improve the system $\boldsymbol{TC}_1$, introducing a new semantic tableaux system for $\boldsymbol{C}_1$, trying to

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[2] $\boldsymbol{C}_1^{1}$, a system stronger than $\boldsymbol{C}_1$, is obtained by replacing the schema of axioms $\neg\neg A\supset A$ of da Costa's $\boldsymbol{C}_1$ by the schema $\neg\neg A\equiv A$.

avoid the presence of infinite loops in the derivations of the branches: in this new tableaux system da Costa's 'ball' operator 'o' is maintained as a defined operator and, as the nodes of the branches in the derivations are not univocally determined, it is possible (as in Buchsbaum and Pequeno's ***SC*1*** system) to occur open branches that may be rebuilt in a distinct way.

Due to the "primitiveness" of the 'ball' operator 'o' and of the other above mentioned denumerable da Costa's operators, in every one of our tableaux systems ***TNDC**$_n$**, $1<n<\omega$, we emphasize that there are specific rules to objectively deal with the operator 'o', as well as with the operators 'k' and '(k)', and the negations '$\sim_k$', for $k\geq 1$. The branches of the tableaux are univocally and automatically generated and infinite loops do not occur.

The systems ***TNDC**$_n$** constitute completely automated theorem proving systems for da Costa's logical systems **C**$_n$*, $1\leq n<\omega$.

As far as we know this is the first paper in the literature in which every one of the operators of the three families of operators 'k' and '(k)', and the negations '$\sim_k$', for $k\geq 1$, introduced as defined operators by da Costa, are considered as primitive operators.

By defining two conditions for the closure of the branches of the tableaux ***TNDC**$_n$**, $1\leq n<\omega$. we look for reflecting the meaning of da Costa's negations in his original hierarchies of paraconsistent systems. This also allows us to capture da Costa's systems ***C**$_n$**, $1\leq n<\omega$, as paraconsistent extensions of classical first-order predicate logic.

Furthermore this is the first paper in which all the systems of da Costa's hierarchy of paraconsistent logics ***C**$_n$**, $1\leq n<\omega$, receive an analytical tableaux approach.

1. Da Costa's paraconsistent first-order logics *C$_n$**

The language **L*** of da Costa's paraconsistent systems ***C**$_n$**, $1\leq n\leq\omega$, has as primitive connectives $\neg$, $\vee$, & and $\supset$ and as primitive quantifiers $\forall$ and $\exists$, as well as a denumerable family of individual variables, predicate symbols and auxiliary symbols (see da Costa (1963a) and (1974)).

The notions of formula, free and bound variables in a formula, sentence, theorem, as well as the general conventions and notations, are the standard ones, as in Kleene (1952).

Let A and B be formulae. As recalled in D'Ottaviano and Castro (2005), the following operators are added, by definition, to the language **L***.

Definition 1.1 $A^o =_{df} \neg(A\&\neg A)$.

Definition 1.2 $A^k =_{def} A^{oo...o}$ ("o" k times, for k ≥1).[3]

We observe that A^1 coincides with A^o.

Definition 1.3 $A^{(k)} =_{df} A^1 \& A^2 \&... \& A^k$, for $k \geq 1$.

Definition 1.4 $\sim_k A =_{df} \neg A\&A^{(k)}$, for $k \geq 1$.

Definition 1.5 $(A\equiv B) =_{df} (A\supset B)\&(B\supset A)$.

In $\boldsymbol{C_n}$*, $1\leq n\leq\omega$, the operator "o" is usually named "*ball operator*" and $A^{(n)}$ may be read as "A is a *well-behaved formula*" or "A is *regular*"; for every $\boldsymbol{C_n}$*, $1\leq n\leq\omega$, the primitive negation "¬" is the basic paraconsistent negation, or *weak negation* of the system, and the connective "$\sim_n$" is called *strong negation*.

For every first-order predicate calculus $\boldsymbol{C_n}$*, $1\leq n<\omega$, the schemata of *axioms* and the *deduction rules* are the following.

Axiom 1: $A\supset(B\supset A)$

Axiom 2: $(A\supset B)\supset((A\supset(B\supset C))\supset(A\supset C))$

Axiom 3: $A\&B\supset A$

Axiom 4: $A\&B\supset B$

Axiom 5: $A\supset(B\supset A\&B)$

Axiom 6: $A\supset A\vee B$

Axiom 7: $A\supset B\vee A$

Axiom 8: $(A\supset C)\supset((B\supset C)\supset(A\vee B\supset C))$

Axiom 9: $\neg\neg A\supset A$

Axiom 10: $A\vee\neg A$

Axiom 11: $B^{(n)}\supset((A\supset B)\supset((A\supset\neg B)\supset\neg A))$

Axiom 12: $A^{(n)}\&B^{(n)}\supset(A\&B)^{(n)}$

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[3] For instance: A^{oo} is $(A^o)^o$, that is, $\neg(A^o\&\neg A^o)$; A^{ooo} is $\neg(A^{oo}\&\neg A^{oo})$; and so on, that is, A^k is $\neg(A^{k-1}\&\neg A^{k-1})$, with $k>1$.

Axiom 13: $A^{(n)}\&B^{(n)}\supset(A\vee B)^{(n)}$

Axiom 14: $A^{(n)}\&B^{(n)}\supset(A\supset B)^{(n)}$

Axiom 15: $\forall xA(x)\supset A(t)$

Axiom 16: $A(t)\supset\exists xA(x)$

Axiom 17: $\forall x(A(x))^{(n)}\supset(\forall xA(x))^{(n)}$

Axiom 18: $\forall x(A(x))^{(n)}\supset(\exists xA(x))^{(n)}$

Axiom 19: A≡B, where A and B are congruent[4] formulae

Rule of *Modus Ponens* (MP)

$$\frac{A,\ A\supset B}{B}$$

Rule II $$\frac{C\supset A(x)}{C\supset\forall xA(x)}$$

Rule III $$\frac{A(x)\supset C}{\exists xA(x)\supset C}\quad ,$$

where the variable x, the term t and the formulae A(x) and C satisfy the usual restrictions[5].

We observe that Axiom 11 is just a particular case of the usual *reductio ad absurdum*, asserting that we can apply the *reductio ad absurdum* in da Costa's paraconsistent logic $\boldsymbol{C_n}^*$ when the formula B is a "well behaved" formula.

Finally, the *system* $\boldsymbol{C_\omega}^*$ is defined by:

Axiom 1 to Axiom 10, Axiom 15-16, Axiom 19 and the Rules MP, II and III.

The classical first-order predicate calculus may be considered as the system $\boldsymbol{C_0}^*$ of this hierarchy. This logic is, in fact, given by

Axiom 1 to Axiom 9, Axiom 15-16, Axiom 19, the Rules MP, II and III, *plus reduction ad absurdum*, that is:

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[4] See Kleene (1952), p. 153.

[5] See Kleene (1952), p. 81.

Axiom 10: $(A\supset B)\supset((A\supset\neg B)\supset\neg A)$.

Following, we present some useful known definitions and results concerning da Costa's paraconsistent systems $\boldsymbol{C_n}$*, most of them necessary for the development of the next sections (see, for instance, Castro (2004) and the recent paper Costa, Krause and Bueno (2006)).

Theorem 1.6 (**Deduction Theorem**) If Γ is a set of formulas, we have that if $\Gamma, A \vdash_{C_n*} B$, then $\Gamma \vdash_{C_n*} A\supset B$; and, if $\Gamma, A \vdash_{C_n*} B$, then $\Gamma \vdash_{C_n*} A\supset B$. □

Theorem 1.7 In $\boldsymbol{C_n}$*, $1\leq n<\omega$, the strong negation $\sim_n$ has all the properties of classical negation. □

Theorem 1.8 Every system in the hierarchy $\boldsymbol{C_n}$*, $1\leq n\leq\omega$, is strictly stronger than those which follow it. □

Definition 1.9 A non-trivial system **S** is said to be *finitely trivializable* if there is a formula (not a schema) F such that, adjoining F to **S** as a new axiom, the resulting system is trivial.

Theorem 1.10 Every paraconsistent system $\boldsymbol{C_n}$*, $1\leq n\leq\omega$, is consistent; the systems $\boldsymbol{C_n}$*, $1\leq n<\omega$, are finitely trivializable; the system $\boldsymbol{C_\omega}$* is not finitely trivializable. □

Theorem 1.11 The systems $\boldsymbol{C_n}$*, $1\leq n\leq\omega$, are undecidable. □

The results of the next theorem are necessary for the development of this work.

Theorem 1.12 In every $\boldsymbol{C_n}$*, $1\leq n<\omega$, we have that:

i. $(A\supset B)\supset(\sim_n A\vee B)$

ii. $A\&\sim_n A\supset B$

iii. $\sim_n\sim_n A\supset A$

iv. $(A\supset B)\supset(\sim_n B\supset\sim_n A)$

v. $(A\vee A)\supset A$

vi. $(\forall x(A(x))^{(n)})\supset(\neg\forall xA(x)\supset\exists x\neg(A(x)))$

vii. $(\forall x(A(x))^{(n)})\supset(\neg\exists x(A(x))\supset\forall x\neg(A(x)))$

viii. $\neg\forall xA(x)\supset(\sim_n(\forall x(A(x))^{(n)})\vee(\exists x\neg(A(x))))$

ix. $\neg\exists x(A(x))\supset(\sim_n(\forall x(A(x))^{(n)})\vee(\forall x\neg(A(x)))$. □

NOTE. – The proofs of (i) and (iv) are immediate consequences of Theorem 1.8; the proof of (v) is immediate; Castro (2004) proved (vi)-(vii); Castro has recently proved (viii)-(ix).

We observe that the Replacement Theorem[6], although valid in $\boldsymbol{C_0}$ and $\boldsymbol{C_0}$* is not valid in general in $\boldsymbol{C_n}$ and $\boldsymbol{C_n}$*, $1\leq n<\omega$.

As, in every system $\boldsymbol{C_n}$*, $1\leq n<\omega$, the formulae $A\supset(\neg A\supset B)$ and $\neg A\supset(A\supset B)$ are not valid, da Costa's systems are paraconsistent systems *lato sensu*, that is, from a contradiction it is not possible in general to deduce any formula.

2. Tableaux systems for C_n*, $1\leq n<\omega$

In this section, we introduce analytical tableaux versions, *à la* Smullyan (1968), for the systems $\boldsymbol{C_n}$*, $1\leq n<\omega$, named ***TNDC****$_n$*. We adapt the notion of tableau sequence presented by van Fraassen (1971).

The language of the systems ***TNDC****$_n$*, $1\leq n<\omega$, is the language **L*** of the logics **C**$_n$*, $1\leq n<\omega$, excepting that we consider the symbol "o" (the ball), the symbols "k", "(k)" and the negations "$\sim_k$", for $k\geq 1$, as primitive symbols. Therefore, the language **L*** contains a (infinite) denumerable set of primitive connectives.

The tableaux method is based on expansion rules, which allow us to analyze the formulae of **L***. Essentially, the expansion rules allow us to expand a sequence of formulae into another sequence of formulae.

We need here to present the following definitions, necessary for the introduction of our hierarchy of tableaux systems, though having them been already introduced by D'Ottaviano and Castro (2005).

Definition 2.1 For every tableaux system ***TNDC****$_n$*, $1\leq n<\omega$, a *tableau sequence* for a given formula S, or simply a *tableau*, is a sequence of expressions A_1, A_2, ... , A_k, such that the formula S is put at the origin of the tableau, as the initial expression A_1; and every expression A_i, $1< i \leq k$, corresponds to a finite disjunction A_i^1 or ... or A_i^m, $m\geq 1$, where every A_i^j, $1\leq j\leq m$, is generated from the preceding expression(s) A_p^j, by applying

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[6] See Kleene (1952), p. 115-116.

one of the Expansion Rules 2.4 of the system. We call each A_i^j a *disjunct* of the expression A_i.

Definition 2.2 For every system ***TNDC***$_n$*, $1 \leq n < \omega$, a *branch* j of a tableau sequence, $1 \leq j \leq m$, corresponds to a sequence of expressions A_i^s, $1 \leq i \leq k$, with A_1^1 the first expression and A_k^j the last one. The superior index s is equal to 1 (s = 1), for $1 \leq i \leq i'$, for some $i' \leq k$; s = j, for $i'' \leq i \leq k$, for some $i'' > i'$; and for $i' < i < i''$, s assumes values between 1 and j.

NOTE. – We observe that the tableau sequence has the structure of a *tree*, if we leave out the disjunction, and write the results of applying any rule under the disjunct to which the rule was applied. Thus, by thinking the disjunction as indicating a branching, the tableau sequence has the structure of an ordered dyadic tree *à la* Smullyan (1968).

For simplicity, the expressions of a given tableau branch j will be identified as of type A_i^j, with $1 \leq i \leq k$ and fixed j, $1 \leq j \leq m$.

Definition 2.3 A *node* corresponds to every expression A_i^j of every branch of a tableau, with $1 \leq i \leq k$ and $1 \leq j \leq m$.

Let the letters α, β, γ, ... , ψ, if necessary also with indexes, stand for formulae of **L***.

Now, suppose that **T** is a tableau being constructed for a initial formula A. Given a certain branch j, let A_{i-1}^j be the last expression of the branch. Then we may extend **T** by one of the following five operations:

(i) If some formula α occurs on the branch of the last expression A_{i-1}^j then, if δ_i^j and δ_{i+1}^j are generated from α by one of the Rules of Conjunctive Type **C** of the system, we may simultaneously adjoin the formulae δ_i^j and δ_{i+1}^j as the next expressions, on the branch j, after A_{i-1}^j;

(ii) If some formula β occurs on the branch of the last expression A_{i-1}^j then, if β_1 or β_2 is generated from β by one of the Rules of Disjunctive Type **D** of the system, we may simultaneously adjoin the formula β_1 to the left of A_{i-1}^j, as the node δ_i^j, and the formula β_2 as the next expression to the right of A_{i-1}^j, as the node δ_i^{j+1};

(iii) If some formula γ occurs on the branch of the last expression A_{i-1}^j then, if the formula δ_i^j is generated from γ by one of the Rules of Special Type $\mathbf{S}_1$ of the system, we may adjoin the formula δ_i^j, on the branch j, as the next expression after A_{i-1}^j;

(iv) If the formulae $\alpha_1, \dots, \alpha_m$ occur on the branch of the last expression A_{i-1}^j then, if the formula δ_i^j is generated from $\{\alpha_1, \dots, \alpha_m\}$ by one of the Rules of Special Type $\mathbf{S}_2$ of the system, we may adjoin, on the branch j, the formula δ_i^j as the next expression after A_{i-1}^j;

(v) If some formula ε occurs on the branch of the last expression A_{i-1}^j then, if the formula δ_i^j is generated from ε by one of the Rules of Special Type $\mathbf{S}_3$ of the system, we may adjoin, on the branch j, the formula δ_i^j as the next expression after A_{i-1}^j;

(vi) If some formula α occurs on the branch of the last expression A_{i-1}^j then, if $\delta_i^j(t)$ is generated from α by one of the Rules of Type **E** of the system, we may adjoin the formulae $\delta_i^j(t)$ as the next expression, on the branch j, after A_{i-1}^j;

(vii) If some formula α occurs on the branch of the last expression A_{i-1}^j then, if $\delta_i^j(c)$ is generated from α by one of the Rules of Type **F** of the system, we may adjoin the formulae $\delta_i^j(c)$ as the next expression, on the branch j, after A_{i-1}^j;

(viii) If some formula β occurs on the branch of the last expression A_{i-1}^j then, if β_1 or β_2 is generated from β by one of the Rules of Type **G** of the system, we may simultaneously adjoin the formula β_1 to the left of A_{i-1}^j, as the node δ_i^j, and the formula β_2 as the next expression to the right of A_{i-1}^j, as the node δ_i^{j+1};

(ix) If some formula γ occurs on the branch of the last expression A_{i-1}^j then, if δ_i^j is generated from γ by one of the Rules of Special Type **H** of the system, we may adjoin the formulae δ_i^j as the next expression, on the branch j, after A_{i-1}^j.

D'Ottaviano and Castro (2005) introduce a hierarchy of syntactical tableaux systems ***TNDC**$_n$*, $1 \leq n < \omega$, in which every system ***TNDC**$_n$* is equivalent to da Costa's corresponding system ***C**$_n$*, $1 \leq n < \omega$. Here, besides the Expansion Rules of ***TNDC**$_n$*, we have specific rules to deal with quantifiers.

The Expansion Rules of ***TNDC**$_n$** are introduced below.

Expansion Rules 2.4 The *expansion rules* of the tableaux systems ***TNDC**$_n$**, $1 \leq n < \omega$, are the following.

2.4.1 *Rules of Conjunctive Type* **C**:

$$\frac{\alpha}{\begin{matrix}\delta_i^j\\ \delta_{i+1}^j\end{matrix}}$$

α	$\delta_i^{\,j}$	δ_{i+1}^j	Name of the Rule
$A\&B$	A	B	$E\&$
$A^{(k)}$	A^k	$A^{(k-1)}$	$E(k)$, $k>1$
$\neg(A^k)$	A^{k-1}	$\neg(A^{k-1})$	$Ek\neg$, $k\geq 1$, where A^0 is A
$\neg(A^{(k)})$	A	$\neg A$	$E(k)\neg$, $k\geq 1$
$\sim_n\neg A$	$\neg\neg A$	$A^{(n)}$	$E\sim_n\neg$
$\sim_n(A^k)$	$\neg(A^k)$	$(A^k)^{(n)}$	$Ek\sim_n$, $k\geq 1$
$\sim_n(A\vee B)$	$\sim_n A$	$\sim_n B$	$DND\sim_n$
$\sim_n(A\supset B)$	A	$\sim_n B$	$DNI\sim_n$
$\sim_n(A^{(k)})$	A	$\neg A$	$E(k)\sim_n$, $k\geq 1$
$\sim_k A$	$\neg A$	$A^{(k)}$	$E\sim_k$, $k<n$ (i)

2.4.2 *Rules of Disjunctive Type* **D**:

$$\frac{\beta}{\delta_i^j \mid \delta_i^{j+1}}$$

β	δ_i^j	δ_i^{j+1}	Name of the rule
$A\vee B$	A	B	$E\vee$
$A\supset B$	$\sim_n A$	B	$E\supset$
$\neg(A\&B)$	$\neg A$	$\neg B$	$DNC\neg$, where B is distinct of $\neg A$ (ii)
$\neg(A\vee B)$	$\neg(A^{(n)}\&B^{(n)})$	$\neg A\&\neg B$	$DND\neg$
$\neg(A\supset B)$	$\neg(A^{(n)}\&B^{(n)})$	$A\&\neg B$	$DNI\neg$
$\sim_n(A\&B)$	$\sim_n A$	$\sim_n B$	$DNC\sim_n$

2.4.3 *Rules of Special Type* **S**$_1$:

$$\frac{\gamma}{\delta_i^j}$$

γ	δ_i^j	Name of the Rule
$\neg\neg A$	A	E$\neg\neg$
$\neg\sim_k A$	A	E$\neg\sim_k$, $k \geq 1$
$\sim_n\sim_k A$	A	E$\sim_n\sim_k$, $k \geq 1$
$\sim_k A$	$\sim_{k-1} A$	R$\sim_k$, $k > n$
A^k	$\neg(A^{k-1}\&\neg A^{k-1})$	Rk, $k\geq1$, where A^0 is A (iii)
$A^{(1)}$	A^1	E(1)

2.4.4 *Rules of Special Type* **S**$_2$:

$$\begin{array}{c}\alpha_1\\ \vdots\\ \alpha_m\\ \hline \delta_i^j\end{array}$$

$\alpha_1, \ldots, \alpha_m$	δ_i^j	Name of the Rule
$\{\neg A, A^1, \ldots, A^k\}$	$\sim_k A$	I$\sim_k$, $0<k<n$
$\{A^1, A^2, \ldots, A^k\}$	$A^{(k)}$	I(k), $0<k<n$ (i)

2.4.5 *Rules of Special Type* **S**$_3$ (iv):

$$\frac{\varepsilon}{\delta_i^j}$$

ε	δ_i^j	Name of the Rule
$A^{oo\ldots o}$	A^k	Eo (with “o” k-times)
$\neg(A^{k-1}\&\neg A^{k-1})$	A^k	Ik, $k\geq1$, where A^0 is A (i)
$(A^s)^k$	A^{s+k}	Is+k, for s, $k \geq 1$
$A^1\&A^2\&\ldots\&A^k$	$A^{(k)}$	I’(k), $k\geq1$ (v)
$\neg A\&A^{(k)}$	$\sim_k A$	I’$\sim_k$, $k\geq1$ (v)

(i) This Rule must be applied only once, on every branch and for every formula.

(ii) If A is of type $(C^{k-1}\&\neg(C^{k-1}))$, then B must be distinct of C^k.

(iii) This Rule must be applied only when there is no possibility of applying any other Rule; it can be applied in subformulae of formulas that occur in the nodes and, in these cases, it must be applied "from outside to inside", that is, from the connective of largest scope to the connective of smallest scope.

(iv) The Rules of Special Type $\mathbf{S}_3$ must be immediately applied, in every case, after applying the first Rule in the initial node of the tableau; they can be applied to subformulae of formulas that occur in the nodes and, in these cases, they must be applied "from outside to inside".

(v) These Rules, under conditions (iv), can only be applied to proper subformulae of formulas that occur in the nodes and, in these cases, they must be applied "from outside to inside".

NOTE. – We observe that A^0, which corresponds to the formula A with superior index "0" (numeral 0), coincides with the formula A. This formula is distinct of the formula A^o ("A-ball").

In the Rules of Special Type $\mathbf{S}_2$ we use the notation of set in order to indicate that it is not important the order in which the formulas occur in the nodes of a branch.

Also, the only Rules that can be applied to subformulae, are the Rules of Special Type $\mathbf{S}_3$ and the Rule Rk.

Following we introduce the specific rules for quantifiers.

2.4.6 *Rules of Type* **E**: $\dfrac{\alpha}{\delta_i^j(t)}$, where t is any term free for any variable occurring in the formula α.

α	$\delta_i^{\,j}$	Name of the Rule
$\forall xA$	$A(t)$	E-$\forall$, with proviso
$\sim_n\exists xA$	$\sim_n(A(t))$	$\sim_n\exists$, with proviso

2.4.7 *Rules of Type* **F** (i): $\dfrac{\alpha}{\delta_i^j(c)}$, where c is a constant that does not occur in the branch; or c has not been previously introduced by Rule of Type **F**, and does not occur in α, and no constant of α has been previously introduced by Rule of Type **F**.

α	$\delta_i^{\ j}$	Name of the Rule
$\exists xA$	$A(c)$	E-$\exists$, with proviso
$\sim_n \forall xA$	$\sim_n(A(c))$	$\sim_n\forall$, with proviso

2.4.8 *Rules of Special Type* **G**: $\dfrac{\beta}{\delta_i^j \mid \delta_i^{j+1}}$

β	δ_i^j	δ_i^{j+1}	Name of the Rule
$\neg\forall xA$	$\sim_n(\forall x(A)^{(n)})$	$\exists x\neg A$	$\neg\forall$
$\neg\exists xA$	$\sim_n(\forall x(A)^{(n)})$	$\forall x\neg A$	$\neg\exists$

2.4.9 *Rules of Special Type* **H**: $\dfrac{\gamma}{\delta_i^j}$

γ	$\delta_i^{\ j}$	Name of the Rule	
A_1	A_2	tA, where $\mathbf{A}_2$ is congruent to $\mathbf{A}_1$	
$\sim_k A_1$	$\sim_k A_2$	sA, $k \geq 1$	(ii)
$\neg A_1$	$\neg A_2$	wA,	(ii)

(i) The Rules of Special Type **F** must be immediately applied, in every case, after applying the first Rule in the initial node of the tableau.

(ii) Where $\mathbf{A}_2$ is congruent to $\mathbf{A}_1$ and $\mathbf{A}_2$ occurs previously in the branch.

NOTE. – In the application of the Expansion Rules, it is more efficient to give priority to the Rules of Type **C**, E-$\exists$, $\sim_n\exists$, $\neg\forall$ and to the Rules of Special Type.

Definition 2.5 For every system ***TNDC**$_n$**, $1\leq n<\omega$, a branch $A_1^j, \ldots , A_s^j$ of a tableau is called a *closed branch* if there exist nodes A_r^j, $1\leq r\leq s$, that correspond either to formulae B and $\sim_n$B, or to formulae B, $\neg$B and $B^1, B^2, \ldots , B^n$.

The next definition gives us the closure criterion for the tableaux.

Definition 2.6 Given a formula S, a *tableau* for S is *closed* if all its branches are closed; otherwise, it is said to be *open*.

Definition 2.7 A *set of formulae* Γ is said to be *closed* if, and only if, there exists a finite subset Γ_0 of Γ, such that there exists a closed tableau for the conjunction of the formulae of Γ_0; otherwise, it is said to be *open*.

In what follows, we use Γ, A as an abbreviation for $\Gamma\cup\{A\}$[7].

Definition 2.8 For every tableaux system ***TNDC**$_n$**, $1\leq n<\omega$, a formula S is said to be an *analytical consequence* of a set Γ of formulae if, and only if, Γ, $\sim_n$S is closed. We also say that Γ, by the Expansion Rules, *generates* S.

This is denoted by: $\Gamma \vdash_{TNDC_n*}$ S.

We observe that, for a formula S to be provable in ***TNDC**$_n$**, $1\leq n<\omega$, a closed tableau must be generated from the strong negation $\sim_n$S of the formula.

Definition 2.9 For every tableaux system ***TNDC**$_n$**, $1\leq n<\omega$, a formula S is said to be *provable* if, and only if, there is a closed tableau for $\sim_n$S, that is, if $\{\sim_n S\}$ is closed.

This is denoted by: $\vdash_{TNDC_n*}$ S.

Definition 2.10 For a given tableau **T** in ***TNDC**$_n$**, $1\leq n<\omega$, a *branch j* is *complete* if, and only if, there is not any Expansion Rule that can still be applied on any node of j.

Definition 2.11 A tableau **T**, in ***TNDC**$_n$**, $1\leq n<\omega$, is *complete* if, and only if, every branch j of **T** is either closed or complete.

---

[7] Γ, A, B is the same as Γ, B, A.

Examples 2.12 Following, we present some examples of proofs in the systems ***TNDC**$_n$****, $1\leq n<\omega$. The rules used are indicated to the right of each step of the proof; the numbers on the left side are added only to facilitate mentioning the tableau.

a) $\nvdash_{\boldsymbol{TNDC}_n}{}^{*}\ \neg\forall x\neg A(x) \equiv \exists x A(x)$

a.1) $\vdash_{\boldsymbol{TNDC}_n}{}^{*}\ \neg\forall x\neg A(x)\supset\exists x A(x)$

1	$\sim_n(\neg\forall x\neg A(x)\supset\exists xA(x))$	
2	$\neg\forall x\neg A(x)$	1, DNI$\sim_n$
3	$\sim_n(\exists xA(x))$	1, DNI$\sim_n$

4	$\sim_n(\forall x(\neg A)^{(n)}))$	2, $\neg\forall$	5	$\exists x\neg\neg A$	2, $\neg\forall$
6	$\sim_n((\neg A(c))^{(n)}))$	4, $\sim_n\forall$	i+1	$\neg\neg A(c)$	5, E-$\exists$
7	$(\neg A(c))^{(n)}$	6, E(k)$\sim_n$	i+2	A(c)	i+1,E$\neg\neg$
8	$\neg((\neg A(c))^{(n)})$	6, E(k)$\sim_n$	i+3	$\sim_n A(c)$	3, $\sim_n\exists$
9	$\neg A(c)$	8, E(k) $\neg$		*	
10	$\neg(\neg A(c))$	8, E(k) $\neg$			
11	A(c)	10, E-$\neg\neg$			
12	$(\neg A(c))^{n}$	7, E(k)			
13	$(\neg A(c))^{(n-1)}$	7, E(k)			
14	$(\neg A(c))^{n-1}$	13, E(k)			
15	$(\neg A(c))^{(n-2)}$	13, E(k)			
16	$(\neg A(c))^{n-2}$	15, E(k)			
17	$(\neg A(c))^{(n-3)}$	15, E(k)			
⋮	⋮				
i-1	$(\neg A(c))^{(1)}$	i-2, E(k)			
i	$(\neg A(c))^{1}$	i-1, E(1)			
	*				

The tableau closes in the first branch by the formulae $\neg A(c)$, $\neg(\neg A(c))$, $(\neg A(c))^{n}$, $(\neg A(c))^{n-1}$, $(\neg A(c))^{n-2}$ .... , $(\neg A(c))^{1}$ that occur on the nodes 9, 10, 11, 14, 16, ... , i; and in the second branch by the formulae A(c) and $\sim_n A(c)$ that occur on the nodes i+2 and i+3.

a.2) $\nvdash_{TNDC_n}* \ \exists xA(x)\supset\neg\forall x\neg A(x)$

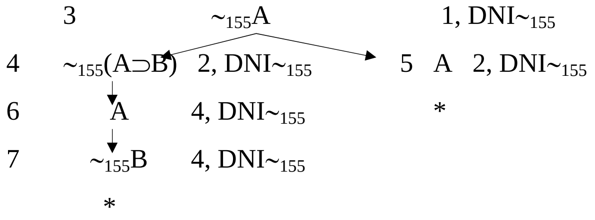
1	$\sim_n(\exists xA(x)\supset\neg\forall x\neg A(x))$	
2	$\exists xA(x)$	1, DNI$\sim_n$
3	$\sim_n(\neg\forall x\neg A(x))$	1, DNI$\sim_n$
4	$A(c)$	2, E-$\exists$
5	$\neg\neg\forall x\neg A(x)$	3, E$\sim_n\neg$
6	$(\forall x\neg A(x))^{(n)}$	3, E$\sim_n\neg$
7	$\forall x\neg A(x)$	5, E $\neg\neg$
9	$\neg A(c)$	7, E $\forall$
10	$(\forall x\neg A(x))^{n}$	6, E(k)
11	$(\forall x\neg A(x))^{(n-1)}$	6, E(k)
12	$(\forall x\neg A(x))^{n-1}$	11, E(k)
13	$(\forall x\neg A(x))^{(n-2)}$	11, E(k)
14	$(\forall x\neg A(x))^{n-2}$	13, E(k)
15	$(\forall x\neg A(x))^{(n-3)}$	13, E(k)
⋮	⋮	
i-1	$(\forall x\neg A(x))^{(1)}$	i-2, E(k)
i	$(\forall x\neg A(x))^{1}$	i-1, E(k)
i+1	$\neg((\forall x\neg A(x))^{n-1}\&\neg(\forall x\neg A(x))^{n-1})$	10, Rk
i+2	$\neg((\forall x\neg A(x))^{n-2}\&\neg(\forall x\neg A(x))^{n-2})$	12, Rk
i+3	$\neg((\forall)\neg A(x))^{n-3}\&\neg(\forall x\neg A(x))^{n-3})$	14, Rk
⋮	⋮	
i+j	$\neg((\forall x\neg A(x))\&\neg(\forall x\neg A(x)))$	i-1, Rk
i+j+1	$(\forall x\neg A(x))^{n}$	i+1, Ik

$i+j+2$	$(\forall x\neg A(x))^{n-1}$	$i+2$, Ik
$i+j+3$	$(\forall x\neg A(x))^{n-2}$	$i+3$, Ik
$i+j+4$	$(\forall x\neg A(x))^{n-3}$	$i+4$, Ik
⋮	⋮	
$i+j+s$	$(\forall x\neg A(x))^{1}$	$i+j$, Ik
$i+j+s+1$	$\neg((\forall x\neg A(x))^{n-1}\&\neg(\forall x\neg A(x))^{n-1})$	$i+j+1$, Rk
$i+j+s+2$	$\neg((\forall x\neg A(x))^{n-2}\&\neg(\forall x\neg A(x))^{n-2})$	$i+j+2$, Rk
$i+j+s+3$	$\neg((\forall)\neg A(x))^{n-3}\&\neg(\forall x\neg A(x))^{n-3})$	$i+j+3$, Rk
⋮	⋮	
$i+j+s+p$	$\neg((\forall x\neg A(x))\&\neg(\forall x\neg A(x)))$	$i+j+s$, Rk
$i+j+s+p+1$	$(\forall x\neg A(x))^{(1)}$	i, I(k)
$i+j+s+p+2$	$(\forall x\neg A(x))^{(2)}$	$i-2$, i, I(k)
$i+j+s+p+2$	$(\forall x\neg A(x))^{(3)}$	$i-4$, i, I(k)
$i+j+s+p+u$	$(\forall x\neg A(x))^{(n-1)}$	12, 14, 16, ..., i, I(k)
⋮	⋮	
$i+j+s+p+u+1$	$(\forall x\neg A(x))^{(n-2)}$	$i+j+s+p+u$, E(k)
$i+j+s+p+u+2$	$(\forall x\neg A(x))^{n-2}$	$i+j+s+p+u+1$, E(k)
$i+j+s+p+u+3$	$(\forall x\neg A(x))^{(n-3)}$	$i+j+s+p+u+2$, E(k)
$i+j+s+p+u+4$	$(\forall x\neg A(x))^{n-3}$	$i+j+s+p+u+3$, E(k)
⋮	⋮	
$i+j+s+p+u+z$	$(\forall x\neg A(x))^{(1)}$	$i+j+s+p+u+z-1$,E(k)
$i+j+s+p+u+z+1$	$(\forall x\neg A(x))^{1}$	$i+j+s+p+u+z$, E(k)

The tableau is complete but is not closed.

b) $\vdash_{\boldsymbol{TNDC}_{155}}$* $((A\supset B)\supset A)\supset A$

1	$\sim_{155}(((A\supset B)\supset A)\supset A)$	
	$\downarrow$	
2	$((A\supset B)\supset A)$	1, DNI$\sim_{155}$
	$\downarrow$	



The tableau is closed by the formulas $\sim_{155}A$ and A, that occur in the nodes 3 and 6 of the first branch, and in the nodes 3 and 5 of the second branch.

3 The Cut Rule for the systems *TNDC$_n$**

In this section we prove a special version of Cut Rule for the systems ***TNDC$_n$****, $1\leq n<\omega$.

Lemma 3.1 If Γ, A is closed, then Γ, A, B is closed.

Proof. Immediate, from Definition 2.7. □

Theorem 3.2 (**Cut Rule**) For every system ***TNDC$_n$****, $1\leq n<\omega$, there exists a closed tableau for a set Γ of formulae if, and only if, for a given formula S there exist closed tableaux either for $\Gamma\cup\{S\}$ and $\Gamma\cup\{\sim_n S\}$, or for $\Gamma\cup\{S\}$ and $\Gamma\cup\{\neg S, S^1, S^2, \ldots, S^n\}$.

Proof. If there exists a closed tableau for Γ, by Lemma 3.1, it is immediate that there are closed tableaux for Γ, S and Γ, $\sim_n S$, and for Γ, S and Γ, $\neg S, S^1, S^2, \ldots, S^n$.

Now, suppose that either there exist closed tableaux for Γ, S and Γ, $\sim_n S$, or there exist closed tableaux for Γ, S and Γ, $\neg S, S^1, S^2, \ldots, S^n$. The proof that there exists a closed tableau for Γ is done by induction on the complexity of the formula S, as we have similarly presented in D’Ottaviano and Castro (2005), Section 4, pp 85-91.

1 Let S be an atomic formula A.

Suppose that there are closed tableaux for Γ, A and Γ, $\sim_n A$. In the cases when either $A\in\Gamma$ or $\sim_n A\in\Gamma$, it is immediate that Γ is closed. Hence, we have only to analyze the case when $A\notin\Gamma$ and $\sim_n A\notin\Gamma$. If either Γ, A or Γ, $\sim_n A$ is closed only on account of formulae of Γ, then Γ is closed and we have nothing to prove; the same reasoning is applicable to the case when we have that Γ, A and Γ, $\neg A, A^1, A^2, \ldots, A^n$ are closed.

1.1 Suppose that there are closed tableaux for Γ, A and for Γ, $\sim_n A$ Observe that from A atomic we can not generate any formula, and from $\sim_n A$ we also can not generate any formula.

If Γ, A is closed then, by Definition 2.7, there is a tableau **T** such that its branches are closed either by $\sim_n A$, or by $\neg A$, A^1, A^2, ... , A^n. As Γ, $\sim_n A$ is also closed, then there is a closed tableau **T'** such that its branches are closed by A, or by $\sim_n \sim_n A$, or by $\neg \sim_n A$ and $(\sim_n A)^1$, $(\sim_n A)^2$, ... , $(\sim_n A)^n$; that is, by Rules $E{\sim_n}{\sim_k}$ and $E\neg{\sim_k}$, the formula A appears in all the branches of **T'**.

Therefore, in the tableaux **T** and **T'** the formulae $\sim_n A$, $\neg A$, A^1, A^2, A^n (in **T**), and A (in **T'**), respectively, are directly generated, by the Expansion Rules from Γ, because, neither $\sim_n A$, $\neg A$, A^1, A^2, A^n could be generated from A, nor A could be generated from $\sim_n A$.

Hence, there is a closed tableau for Γ and so, by Definition 2.7, Γ is closed.

1.2 Suppose that there are closed tableaux for Γ, A and for Γ, $\neg A$, A^1, A^2, ... , A^n. Observe that it is not possible to generate any formula from A and from $\neg A$, A^1, A^2, ... , A^n it is only possible to generate $\sim_k A$ and $A^{(k)}$, $k<n$ (by Rules $I{\sim_k}$ and I(k)).

If Γ, A is closed then, by Definition 2.7, there is a tableau **T** such that its branches are closed either by $\sim_n A$, or by $\neg A$ and A^1, A^2, ... , A^n.

As Γ, $\neg A$, A^1, A^2, ... , A^n is also closed, then there is a closed tableau **T'** such that its branches are closed by A; or by $\sim_n \neg A$; or by $\neg\neg A$, $(\neg A)^1$, $(\neg A)^2$, ... , $(\neg A)^n$; or by $\sim_n(A^i)$, for every i, $1 \leq i \leq n$; or by $\neg(A^i)$, $(A^i)^1$, $(A^i)^2$, ... , $(A^i)^n$, for every i, $1 \leq i \leq n$. So, by Rules $E{\sim_n}\neg$, $E\neg\neg$, $Ek{\sim_n}$, E& and $Ek\neg$, the formula A appears in all the branches of **T'**.

Therefore, in the tableaux **T** and **T'** the formulae $\sim_n A$ or $\neg A$, A^1, A^2, ... , A^n (in **T**) and A (in **T'**), respectively, are directly generated, by the Expansion Rules, from Γ.

Hence, there exists a closed tableau for Γ and so, by Definition 2.7, Γ is closed.

2 Suppose that the result holds for formulae S of complexity p, $p>0$.

3 Let S be a formula of complexity p+1.

The cases where S is of type $\neg B$, with B of complexity p; S is of type B^k, $k \geq 1$; S is of type $B^{(k)}$, with $k \geq 1$; S is of type $\sim_k B$, with $k \geq 1$: S is of type $(B \& C)$; S is of type $(B \vee C)$ and S is of type $(B \supset C)$ can be proved exactly as in D´Ottaviano and Castro (2005).

3.1 Let S be of type $\forall x B$, with B of complexity p.

3.1.1 Let Γ, $\forall xB$ and Γ, $\sim_n\forall xB$ be closed, considering that $\forall xB$ and $\sim_n\forall xB$ are not formulae of Γ.

.If Γ, $\sim_n\forall xB$ is closed, then, by Rule $\sim_n\forall$, Γ, $\sim_n B(c)$ is closed, with c not occurring in the considered branch (with c not occurring in the considered branch of the tableaux; or c has not been previously introduced by Rule of Type **F**, and does not occur in $\exists xB$, and no constant of $\exists xB$ has been previously introduced by Rule of Type **F**).

As Γ, $\forall xB$ is also closed then, by Rule E-$\forall$, Γ, B(c) is closed.

So, as Γ, B(c) and Γ, $\sim_n$ B(c) are closed, hence by induction hypothesis, Γ is closed.

3.1.2 Let Γ, $\forall xB$ and Γ, $\neg\forall xB$, $(\forall xB)^1$, $(\forall xB)^2$, ... , $(\forall xB)^n$ be closed, also considering that $\forall xB$, $\neg\forall xB$ e $(\forall xB)^i$, for every i, $1\leq i\leq n$, are not formulae of Γ. We observe that, from $((\forall xB)^i$, for $1\leq i\leq k$, by Rulr Rk, $k\geq1$, it is only possible to generate the formula $\neg((\forall xB)^{i-1}\&\neg((\forall xB)^{i-1}))$.

If Γ, $\neg\forall xB$, $(\forall xB)^1$, $(\forall xB)^2$, ..., $(\forall xB)^n$ is closed then, by Rule $\neg\forall$, there exist closed tableaux for Γ, $\sim_n(\forall x(B)^{(n)})$, $(\forall xB)^1$, $(\forall xB)^2$, ..., $(\forall xB)^n$ and for Γ, $\exists x\neg B$, $(\forall xB)^1$, $(\forall xB)^2$, ..., $(\forall xB)^n$. Therefore,

(i) Γ, $\sim_n(\forall x(B)^{(n)})$, $(\forall xB)^1$, $(\forall xB)^2$, ..., $(\forall xB)^n$ is closed, that is, Γ, $(\forall xB)^1$, $(\forall xB)^2$, ..., $(\forall xB)^n$ $\sim_n(\forall x(B)^{(n)})$ is closed.

As Γ, $\forall xB$ is also closed, by Lemma 3.1, we have that Γ, $\forall xB$, $(\forall xB)^1$, $(\forall xB)^2$, ..., $(\forall xB)^n$, $\forall x(B)^{(n)}$ is closed.

Hence, by induction hypothesis, we have that Γ, $\forall xB$, $(\forall xB)^1$, $(\forall xB)^2$, ..., $(\forall xB)^n$ is closed.

(ii) If Γ, $\exists x\neg B$, $(\forall xB)^1$, $(\forall xB)^2$, ..., $(\forall xB)^n$ is closed, then, by Rule E-$\exists$, Γ, $\neg B(c)$, $(\forall xB)^1$, $(\forall xB)^2$, ..., $(\forall xB)^n$ is closed, with c not occurring in the considered branch of the tableaux; or c has not been previously introduced by Rule of Type **F**, and does not occur in $\exists x\neg B$, and no constant of $\exists x\neg B$ has been previously introduced by Rule of Type **F**. That is, Γ, $(\forall xB)^1$, $(\forall xB)^2$, ..., $(\forall xB)^n$, $\neg B(c)$ is closed.

(iii) As, by (i), Γ, $\forall x(B)$, $(\forall xB)^1$, $(\forall xB)^2$, ..., $(\forall xB)^n$ is closed then, by Rule E-$\forall$, Γ, B(c), $(\forall xB)^1$, $(\forall xB)^2$, ..., $(\forall xB)^n$ is also closed, that is, Γ, $(\forall xB)^1$, $(\forall xB)^2$, ..., $(\forall xB)^n$, B(c) is closed. So, by (ii), by induction hypothesis, Γ, $(\forall xB)^1$, $(\forall xB)^2$, ..., $(\forall xB)^n$ is closed.

If there is a branch that is closed on account of Γ and anyone of the formulae $(\forall xB)^1$, $(\forall xB)^2$, ..., $(\forall xB)^n$, then it is closed either by $\sim_n((\forall xB)^i)$; or by $\neg((\forall xB)^i)$, $((\forall xB)^i)^1$, $((\forall xB)^i)^2$,…, $((\forall xB)^i)^{n}$, for every, $1 \leq i \leq n$. Then, by applying Rules Ek$\sim_n$, E& and Ek$\neg$, in these previous mentioned formulae, the formula $\forall xB$ appears in this branch. As $\forall xB$ can not be generated from $\neg\forall xB$, $\sim_n(\forall x(B)^{(n)})$ or $\exists x\neg B$, by the Expansion Rules, Γ

generates $\forall xB$. Therefore, as Γ, $\forall xB$ is closed and Γ generates $\forall xB$, we have that Γ is closed.

3.2 Let S be of type $\exists xB$, with B of complexity p.

3.2.1. Let Γ, $\exists xB$ and Γ, $\sim_n\exists xB$ be closed, considering that $\exists xB$ and $\sim_n\exists xB$ are not formulae of Γ.

If Γ, $\exists xB$ is closed, then, by Rule E-$\exists$, Γ, B(c) is closed (with c not occurring in the considered branch of the tableaux; or c has not been previously introduced by Rule of Type **F**, and does not occur in $\exists xB$, and no constant of $\exists xB$ has been previously introduced by Rule of Type **F**).

If Γ, $\sim_n\exists xB$ is closed, then. by Rule $\sim_n\exists$, Γ, $\sim_n B(c)$ is closed.

Hence, as Γ, B(c) and Γ, $\sim_n B(c)$ are closed, then by induction hypothesis, Γ is closed.

3.2.2 Let Γ, $\exists xB$ and Γ, $\neg\exists xB$, $(\exists xB)^1$, $(\exists xB)^2$, ... , $(\exists xB)^n$ be closed, considering also that $\exists xB$, $\neg\exists xB$ and $(\exists xB)^i$, for any i, $1\leq i\leq n$, are not formulae of Γ. We observe that, from $(\exists xB)^{i}$, for $1\leq i\leq k$, by Rule Rk, $k\geq 1$, it is only possible to generate the formula $\neg((\exists xB)^{i-1}\&\neg((\exists xB)^{i-1}))$.

If Γ, $\exists xB$ is closed, then, by Rule E-$\exists$, Γ, B(c) is closed, with proviso.

If Γ, $\neg\exists xB$, $(\exists xB)^1$, $(\exists xB)^2$, ..., $(\exists xB)^n$ is closed then, by Rule $\neg\exists$, there exist closed tableaux for Γ, $\sim_n(\forall x(B)^{(n)})$, $(\exists xB)^1$, $(\exists xB)^2$, ..., $(\exists xB)^n$ and for Γ, $\forall x\neg B$, $(\exists xB)^1$, $(\exists xB)^2$, ..., $(\exists xB)^n$. Therefore,

(i) Γ, $\sim_n(\forall x(B)^{(n)})$, $(\exists xB)^1$, $(\exists xB)^2$, ..., $(\exists xB)^n$ is closed, by Lemma 3.1, we have that, Γ, $\exists xB$, $(\exists xB)^1$, $(\exists xB)^2$, ..., $(\exists xB)^n$ $\sim_n(\forall x(B)^{(n)})$ is closed.

As Γ, $\exists xB$ is also closed, again by Lemma 3.1, we have that Γ, $\exists xB$, $(\exists xB)^1$, $(\exists xB)^2$, ..., $(\exists xB)^n$, $\forall x(B)^{(n)}$ is closed.

Hence, by induction hypothesis, we have that Γ, $\exists xB$, $(\exists xB)^1$, $(\exists xB)^2$, ..., $(\exists xB)^n$ is closed. So, by applying Rule E-$\exists$, Γ, B(c), $(\exists xB)^1$, $(\exists xB)^2$, ..., $(\exists xB)^n$ is closed (with proviso). That is, Γ, $(\exists xB)^1$, $(\exists xB)^2$, ..., $(\exists xB)^n$, B(c) is closed.

(ii) If Γ, $\forall x\neg B$, $(\exists xB)^1$, $(\exists xB)^2$, ..., $(\exists xB)^n$ is closed, then, by Rule E-$\forall$, Γ, $\neg B(c)$, $(\exists xB)^1$, $(\exists xB)^2$, ..., $(\exists xB)^n$ is closed, with proviso. That is, Γ, $(\exists xB)^1$, $(\exists xB)^2$, ..., $(\exists xB)^n$, $\neg B(c)$ is closed and so, by Lemma 3.1, Γ, $(\exists xB)^1$, $(\exists xB)^2$, ..., $(\exists xB)^n$, $\neg B(c)$, $(B(c))^1$, $(B(c))^2$, ..., $(B(c))^n$ is closed.

(iii) As, by (i) and (ii), Γ, $(\exists xB)^1$, $(\exists xB)^2$, ..., $(\exists xB)^n$, B(c) and Γ, $(\exists xB)^1$, $(\exists xB)^2$, ..., $(\exists xB)^n$, $\neg$B(c), $(B(c))^1$, $(B(c))^2$, ..., $(B(c))^n$ are closed then, by induction hypothesis, Γ, $(\exists xB)^1$, $(\exists xB)^2$, ..., $(\exists xB)^n$ is closed.

If there is a branch that is closed on account of Γ and anyone of the formulae $(\exists xB)^1$, $(\exists xB)^2$, ..., $(\exists xB)^n$, then it is closed by $\sim_n ((\exists xB)^i)$; or by $\neg((\exists xB)^i)$, $((\exists xB)^i)^1$, $((\exists xB)^i)^2$,…, $((\exists xB)^i)^n$, for every, $1 \leq i \leq n$. Then, by Rules Ek$\sim_n$, E& and Ek$\neg$, the formula $\exists$xB appears in this branch. As $\exists$xB can not be generated from $\neg\exists$xB or $\sim_n(\exists x(B)^{(n)})$ or $\forall x\neg$B, by the Expansion Rules, Γ generates $\exists$xB. Therefore, as Γ, $\exists$xB is closed and Γ generates $\exists$xB, we have that Γ is closed.

Hence, by Cases 1-3, we have proved that if either Γ, S and Γ, $\sim_n$S are closed, or Γ, S and Γ, S, $\neg$S, S^1, S^2, ... , S^n are closed, then Γ is closed. □

4 The logical equivalence between the systems of the hierarchy *TNDC$_n$** and the corresponding da Costa's systems *C$_n$*, 1≤n<ω*

Now, based on the Cut Rule for ***TNDC$_n$****, *1≤n<ω*, we can prove the equivalence between the systems ***TNDC$_n$**** and the corresponding da Costa's paraconsistent systems ***C$_n$****, *1≤n<ω*.

Theorem 4.1 If $\Gamma \vdash_{C_n^*}$ S, then $\Gamma \vdash_{TNDC_n^*}$ S, for every n, $1 \leq n < \omega$.

Proof. Suppose that $\Gamma \vdash_{C_n^*}$ S.

If S$\in\Gamma$ then, for every n, $1 \leq n < \omega$, it is immediate that $\Gamma \vdash_{TNDC_n^*}$ S. So, let us suppose that S is not in Γ.

1 Let S be an axiom schema of ***C$_n$****, *1≤n<ω*.

Let us prove that $\Gamma \vdash_{TNDC_n^*}$ S, that is, we have to prove that Γ, $\sim_n$S is closed in ***TNDC$_n$****, *1≤n<ω*.

Here, we present the proofs for the Axiom schemata 15 to 19 and for the Deduction Rules. The cases of Axioms 1 to 14 were proved by D´Ottaviano and Castro (2005).

1.1 Let S be Axiom 15, that is, S is $\forall xA\supset A(t)$, where t and the formulae A(x) satisfy the usual restrictions[8]. We shall generate a closed tableau, whose initial node, $\Gamma, \sim_n S$, constitutes the step 1 below.

1	$\Gamma, \sim_n(\forall xA\supset A(t))$	
2	$\Gamma, \forall xA$	1, DNI$\sim_n$
3	$\Gamma, \sim_n(A(t))$	1, DNI$\sim_n$
4	$\Gamma, A(t)$	2, E$\forall$
	*	

In this case, the tableau closes by the formulae $\sim_n(A(t))$ and A(t), occurring in the nodes 3 and 4, respectively.

1.2 Let S be Axiom 16, that is, S is $A(t) \supset(\exists xA)$, where t is a term which is free for x in A(x). We shall generate a closed tableau, whose initial node, $\Gamma, \sim_n S$, constitutes the step 1 below.

1	$\Gamma, \sim_n(A(t) \supset(\exists xA))$	
2	$\Gamma, A(t)$	1, DNI$\sim_n$
3	$\Gamma, \sim_n(\exists xA)$	1, DNI$\sim_n$
4	$\Gamma, \sim_n(A(t))$	3, $\sim_n\exists$
	*	

1.3 Let S be Axiom 17, that is, S is $\forall x(A)^{(n)}\supset(\forall xA)^{(n)}$.

1	$\Gamma, \sim_n(\forall x(A)^{(n)}\supset(\forall xA)^{(n)})$	
2	$\Gamma, \forall x(A)^{(n)}$	1, DNI$\sim_n$
3	$\Gamma, \sim_n((\forall xA)^{(n)})$	1, DNI$\sim_n$
4	$\Gamma, (\forall xA)$	3, E(k)$\sim_n$
5	$\Gamma, \neg(\forall xA)$	3, E(k)$\sim_n$

6	$\Gamma, \sim_n(\forall x(A)^{(n)})$	5, $\neg\forall$	7	$\Gamma, \exists x\neg A$	5, $\neg\forall$
	*		8	$\Gamma, \neg A(c)$	7, E-$\exists$
			9	$\Gamma, A(c)$	4, E-$\forall$

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[8] See Kleene (1952), p. 81.

10	$\Gamma, (A(c))^{(n)}$	2, E-$\forall$
11	$\Gamma, (A(c))^{n}$	10, E(k)
12	$\Gamma, (A(c))^{(n-1)}$	10, E(k)
13	$\Gamma, (A(c))^{n-1}$	12, E(k)
14	$\Gamma, (A(c))^{(n-2)}$	12, E(k)
15	$\Gamma, (A(c))^{n-2}$	14, E(k)
⋮	⋮	
i-1	$\Gamma, (A(c))^{(1)}$	i-2, E(k)
i	$\Gamma, (A(c))^{1}$	i-1, E(1)
	*	

1.4 Let S be Axiom 18, that is, S is $\forall x(A)^{(n)} \supset (\exists xA)^{(n)}$.

1	$\Gamma, \sim_n(\forall x(A)^{(n)} \supset (\exists xA)^{(n)})$	
2	$\Gamma, \forall x(A)^{(n)}$	1, DNI$\sim_n$
3	$\Gamma, \sim_n((\exists xA)^{(n)})$	1, DNI$\sim_n$
4	$\Gamma, (\exists xA)$	3, E(k)$\sim_n$
5	$\Gamma, \neg(\exists xA)$	3, E(k)$\sim_n$

6	$\Gamma, \sim_n(\forall x(A)^{(n)})$	5, $\neg\exists$	7	$\Gamma, \forall x \neg A$	5, $\neg\exists$
	*		8	$\Gamma, A(c)$	4, E-$\exists$
			9	$\Gamma, \neg A(c)$	7, E-$\forall$
			10	$\Gamma, (A(c))^{(n)}$	2, E-$\forall$
			11	$\Gamma, (A(c))^{n}$	10, E(k)
			12	$\Gamma, (A(c))^{(n-1)}$	10, E(k)
			13	$\Gamma, (A(c))^{n-1}$	12, E(k)
			14	$\Gamma, (A(c))^{(n-2)}$	12, E(k)
			15	$\Gamma, (A(c))^{n-2}$	14, E(k)
			⋮	⋮	
			i-1	$\Gamma, (A(c))^{(1)}$	i-2, E(k)

i	$\Gamma, (A(c))^1$	i-1, E(1)
	*	

1.5 Let S be Axiom 19, that is, S is $A \equiv B$, where A and B are congruent formulae.

Immediate, from Rules E-$\forall$, $\sim_n\forall$, E-$\exists$, $\sim_n\exists$ and Special Rules **H**.

2 Now, let us consider that the formula S is a consequence of preceding formulae in a proof in $\boldsymbol{C}_n$*, $1 \leq n < \omega$, by *Modus Ponens*; that is, we have that $\Gamma \vdash_{C_n^*}$ S is a consequence of $\Gamma \vdash_{C_n^*}$ A and $\Gamma \vdash_{C_n^*} A \supset S$. Then, as we have that $\Gamma \vdash_{TNDC_n^*}$ A and $\Gamma \vdash_{TNDC_n^*} A \supset S$, the sets $\Gamma \cup \{\sim_n A\}$ and $\Gamma \cup \{\sim_n (A \supset S)\}$ are closed in ***TNDC**$_n$** and so, by Rule DNI$\sim_n$, $\Gamma \cup \{\sim_n A\}$ and $\Gamma \cup \{A, \sim_n S\}$ are closed. Hence, $\Gamma, \sim_n S$, A and $\Gamma, \sim_n S, \sim_n A$ are closed and, so, by the Cut Rule, $\Gamma, \sim_n S$ is closed. Therefore, Γ generates S in ***TNDC**$_n$**, $1 \leq n < \omega$, that is, $\Gamma \vdash_{TNDC_n^*}$ S.

3 Let us consider that the formula S is a consequence of preceding formula in a proof in $\boldsymbol{C}_n$*, $1 \leq n < \omega$, by an application of Rule II; that is, we have that $\Gamma \vdash_{C_n^*} C \supset \forall x A(x)$ is a consequence of $\Gamma \vdash_{C_n^*} C \supset A(x)$, where C[9] is a formula which does not contain x free. Let us suppose that $\Gamma \nvdash_{TNDC_n^*} C \supset \forall x A(x)$, then, as we have that $\Gamma \vdash_{TNDC_n^*} C \supset A(x)$, the set $\Gamma \cup \{\sim_n (C \supset A(x))\}$ is closed and $\Gamma \cup \{\sim_n (C \supset \forall x A(x))\}$ is not closed in ***TNDC**$_n$**; and so, by Rule DNI$\sim_n$, $\Gamma \cup \{C, \sim_n A(x)\}$ is closed and $\Gamma \cup \{C, \sim_n (\forall x A(x))\}$ is not closed and so, by Rule $\sim_n\forall$, $\Gamma \cup \{C, \sim_n A(c)\}$ is not closed, where c is a constant that does not occur in the branch; or c has not been previously introduced by Rule of Type **F**, and does not occur in $\forall x A(x)$, and no constant of $\forall x A$ has been previously introduced by Rule of Type **F**. From $\Gamma \cup \{C, \sim_n A(x)\}$ is closed, we obtain $\Gamma \cup \{C, \sim_n A(c)\}$[10] is closed. Hence, we obtain that $\Gamma \cup \{C, \sim_n A(c)\}$ is closed and $\Gamma \cup \{C, \sim_n A(c)\}$ is not closed.

4 Let us consider that the formula S is a consequence of preceding formula in a proof in **C**$_n$*, $1 \leq n < \omega$, by an application of Rule III; that is, we have that $\Gamma \vdash_{C_n^*} \exists x A(x) \supset C$ is a consequence of $\Gamma \vdash_{C_n^*} A(x) \supset C$, where C is a formula which does not contain x free. Let us suppose that $\Gamma \nvdash_{TNDC_n^*} \exists x A(x) \supset C$; then, as we have that $\Gamma \vdash_{TNDC_n^*} A(x) \supset C$, the set $\Gamma \cup \{\sim_n (A(x) \supset C)\}$ is closed and $\Gamma \cup \{\sim_n (\exists x A(x) \supset C)\}$ is not closed in ***TNDC**$_n$**; and so, by Rules DNI$\sim_n$ and E-$\exists$, $\Gamma \cup \{A(x), \sim_n C\}$ is closed and $\Gamma \cup \{A(c), \sim_n C\}$ is not closed. Hence, there exists at least one constant c (in the domain), such that $\Gamma \cup \{A(x), \sim_n C\}$ is not closed, but it is an absurd. □

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[9] We observe that x, in $C \supset A(x)$, denotes necessarily any constant in the domain.

[10] Where c is a constant that does not occur in the branch.

Remark 4.2 The Deduction Theorem is provable in the ***TNDC**$_n$**, $1 \leq n < \omega$, with the necessary restrictions, as in ***TNDC**$_n$*, $1 \leq n < \omega$.[11] □

Theorem 4.3 If $\Gamma \vdash_{TNDC_n*} S$, then $\Gamma \vdash_{C_n*} S$.

Proof. Suppose that $\Gamma \vdash_{TNDC_n*} S$. If $S \in \Gamma$, then $\Gamma \vdash_{C_n*} S$ is immediate. So, let us suppose that S is not in Γ.

In order to prove the theorem, let us consider S as a formula generated from Γ by the expansion rules of ***TNDC**$_n$**, $1 \leq n < \omega$.

We shall transform every Expansion Rule of ***TNDC**$_n$**, $1 \leq n < \omega$, into a correspondent valid proof in ***C**$_n$**, $1 \leq n < \omega$. That is, the rules of type C, D, S_1, S_2, S_3, E, F, G and H will be transformed into the proofs of $\alpha \vdash_{C_n*} (\delta_i^j)\&(\delta_{i+1}^j)$; $\beta \vdash_{C_n*} (\delta_i^j)\vee(\delta_{i+1}^j)$; $\gamma \vdash_{C_n*} \delta_i^j$; $\alpha_1,\ldots, \alpha_n \vdash_{C_n*} \delta_i^j$; $\varepsilon \vdash_{C_n*} \delta_i^j$, $\alpha \vdash_{C_n*} \delta_i^j(t)$; $\alpha \vdash_{C_n*} \delta_i^j(c)$; $\beta \vdash_{C_n*} (\delta_i^j)\vee(\delta_{i+1}^j)$ and $\gamma \vdash_{C_n*} \delta_i^j$, respectively.

We shall only present the complete proofs relative to the Expansion Rules involving the quantifiers. The other cases of Expansion Rules were proved by D´Ottaviano and Castro (2005).

1. Let S be of type A(t), generated, in ***TNDC**$_n$**, $1 \leq n < \omega$, from $\forall xA(x)$, by Rule E-$\forall$, where t is any term free for any variable occurring in the formula $\forall xA(x)$. We have to prove that $\forall xA \vdash_{C_n*} A(t)$.

From Axiom 15 and the Deduction Theorem, the proof is immediate.

2. Let S be of type A(c), generated, in ***TNDC**$_n$**, $1 \leq n < \omega$, from $\exists xA(x)$, by Rule E-$\exists$, with c not occurring in the considered branch of the tableaux; or c has not been previously introduced by Rule of Type **F**, and does not occur in $\exists xA(x)$, and no constant of $\exists xA$ has been previously introduced by Rule of Type **F**. We have to prove that $\exists xA(x) \vdash_{C_n*} A(c)$.

1. $A(c) \vdash_{C_n*} A(c)$	property of $\vdash_{C_n*}$
2. $\vdash_{C_n*} A(c) \supset A(c)$	1, Deduction Theorem
3. $\vdash_{C_n*} \exists xA(x) \supset A(c)$	2, Rule III
4. $\exists xA(x) \vdash_{C_n*} \exists xA(x)$	property of $\vdash_{C_n*}$

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[11] D´Ottaviano and Castro (2005) prove the Deduction Theorem for the ***TDNC**$_n$*, $1 \leq n < \omega$.

5. $\exists xA(x) \vdash_{C_n*} A(c)$	3, 4, MP

3. Let S be of type $\sim_n A(t)$, generated, in ***TNDC$_n$*****, $1\leq n<\omega$, from $\sim_n\exists xA(x)$, by Rule $\sim_n\exists$, where t is any term free for any variable occurring in the formula $\exists xA(x)$.We have to prove that $\sim_n\exists xA(x) \vdash_{C_n*} \sim_n A(t)$.

1	$\sim_n\exists xA(x), A(x) \vdash_{C_n*} \sim_n\exists xA(x)$	property of $\vdash_{C_n*}$
2	$\sim_n\exists xA(x), A(x) \vdash_{C_n*} A(x)$	property of $\vdash_{C_n*}$
3	$\vdash_{C_n*} A(x)\supset\exists xA(x)$	Axiom 16
4	$\sim_n\exists xA(x), A(x) \vdash_{C_n*} \exists xA(x)$	2, 3, MP
5	$\sim_n\exists xA(x), A(x) \vdash_{C_n*} \exists xA(x)\supset(\sim_n\exists xA(x)\supset(\exists xA(x)\&\sim_n\exists xA(x)))$	Axiom 5
6	$\sim_n\exists xA(x), A(x) \vdash_{C_n*} \sim_n\exists xA(x)\supset(\exists xA(x)\&\sim_n\exists xA(x))$	4, 5, MP
7	$\sim_n\exists xA(x), A(x) \vdash_{C_n*} \exists xA(x)\&\sim_n\exists xA(x)$	1, 6, MP
8	$\vdash_{C_n*} \exists xA(x)\&\sim_n\exists xA(x)\supset\sim_n A(x)$	Theorem 1.12 (ii)
9	$\sim_n\exists xA(x), A(x) \vdash_{C_n*} \sim_n A(x)$	7, 8, MP
10	$\sim_n\exists xA(x) \vdash_{C_n*} A(x)\supset\sim_n A(x)$	9, Deduction Theorem
11	$\vdash_{C_n*} (A(x)\supset\sim_n A(x))\supset(\sim_n A(x)\vee\sim_n A(x))$	Theorem 1.12 (i)
12	$\sim_n\exists xA(x) \vdash_{C_n*} \sim_n A(x)\vee\sim_n A(x)$	10, 11, MP
13	$\vdash_{C_n*} \sim_n A(x)\vee\sim_n A(x)\supset\sim_n A(x)$	Theorem 1.12 (v)
14	$\sim_n\exists xA(x) \vdash_{C_n*} \sim_n A(x)$	12, 13, MP
15	$\vdash_{C_n*} \sim_n\exists xA(x)\supset\sim_n A(x)$	14, Deduction Theorem
16	$\vdash_{C_n*} \sim_n\exists xA(x)\supset\forall x\sim_n A(x)$	15, Rule II
17	$\sim_n\exists xA(x) \vdash_{C_n*} \sim_n\exists xA(x)$	property of $\vdash_{C_n*}$
18	$\sim_n\exists xA(x) \vdash_{C_n*} \forall x\sim_n A(x)$	16, 17, MP
19	$\vdash_{C_n*} \forall x\sim_n A(x)\supset\sim_n A(t)$	Axiom 15
20	$\sim_n\exists xA(x) \vdash_{C_n*} \sim_n A(t)$	18, 19, MP

4. Let S be of type $\sim_n A(c)$, generated, in ***TNDC$_n$*****, $1\leq n<\omega$, from $\sim_n\forall xA(x)$, by Rule $\sim_n\forall$, with c not occurring in the considered branch of the tableaux; or c has not been previously introduced by Rule of Type **F**, and does not occur in $\forall xA(x)$, and no constant

of $\forall xA(x)$ has been previously introduced by Rule of Type **F**. We have to prove that $\sim_n\forall xA(x) \vdash_{C_n^*} \sim_n A(c)$.

1. $\sim_n\exists x \sim_n A(x),\ \sim_n A(x) \vdash_{C_n^*} \sim_n\exists x\sim_n A(x)$ property of $\vdash_{C_n^*}$
2. $\sim_n\exists x\sim_n A(x),\ \sim_n A(x) \vdash_{C_n^*} \sim_n A(x)$ property of $\vdash_{C_n^*}$
3. $\vdash_{C_n^*} \sim_n A(x)\supset\exists x\sim_n A(x)$ Axiom 16
4. $\sim_n\exists x\sim_n A(x), \sim_n A(x) \vdash_{C_n^*} \exists x\sim_n A(x)$ 2, 3, MP
5. $\sim_n\exists x\sim_n A(x), \sim_n A(x) \vdash_{C_n^*} \exists x\sim_n A(x)\supset(\sim_n\exists x\sim_n A(x) \supset(\exists x\sim_n A(x)\&\sim_n\exists x\sim_n A(x)))$

   Axiom 5
6. $\sim_n\exists x\sim_n A(x), \sim_n A(x) \vdash_{C_n^*} \sim_n\exists x\sim_n A(x)\supset(\exists x\sim_n A(x)\&\sim_n\exists x\sim_n A(x))$ 4, 5, MP
7. $\sim_n\exists x\sim_n A(x),\ \sim_n A(x) \vdash_{C_n^*} \exists x\sim_n A(x)\&\sim_n\exists x\sim_n A(x)$ 1, 6, MP
8. $\vdash_{C_n^*} \exists x\sim_n A(x)\&\sim_n\exists x\sim_n A(x)\supset\sim_n\sim_n A(x)$ Theorem 1.12 (ii)
9. $\sim_n\exists x\sim_n A(x),\ \sim_n A(x) \vdash_{C_n^*} \sim_n\sim_n A(x)$ 7, 8, MP
10. $\sim_n\exists x\sim_n A(x) \vdash_{C_n^*} \sim_n A(x)\supset\sim_n\sim_n A(x)$ 9, Deduction Theorem
11. $\vdash_{C_n^*} (\sim_n A(x)\supset\sim_n\sim_n A(x))\supset(\sim_n\sim_n A(x)\vee\sim_n\sim_n A(x))$ Theorem 1.12 (i)
12. $\sim_n\exists x\sim_n A(x) \vdash_{C_n^*} \sim_n\sim_n A(x)\vee\sim_n\sim_n A(x)$ 10, 11, MP
13. $\vdash_{C_n^*} \sim_n\sim_n A(x)\vee\sim_n\sim_n A(x)\supset\sim_n\sim_n A(x)$ Theorem 1.12 (ix)
14. $\sim_n\exists x\sim_n A(x) \vdash_{C_n^*} \sim_n\sim_n A(x)$ 12, 13, MP
15. $\vdash_{C_n^*} \sim_n\sim_n A(x)\supset A(x)$ Theorem 1.12 (iii)
16. $\sim_n\exists x\sim_n A(x) \vdash_{C_n^*} A(x)$ 14, 15, MP
17. $\vdash_{C_n^*} \sim_n\exists x\sim_n A(x)\supset A(x)$ 16, Deduction Theorem
18. $\vdash_{C_n^*} \sim_n\exists x\sim_n A(x)\supset\forall xA(x)$ 17, Rule II
19. $\vdash_{C_n^*} (\sim_n\exists x\sim_n A(x)\supset\forall xA(x))\supset(\sim_n\forall xA(x)\supset\sim_n\sim_n\exists x\sim_n A(x))$ Theorem 1.12 (iv)
20. $\vdash_{C_n^*} \sim_n\forall xA(x)\supset\sim_n\sim_n\exists x\sim_n A(x)$ 18, 19, MP
21. $\vdash_{C_n^*} \sim_n\sim_n\exists x\sim_n A(x)\supset\exists x\sim_n A(x)$ Theorem 1.12 (iii)
22. $\vdash_{C_n^*} \sim_n\forall xA(x)\supset\exists x\sim_n A(x)$ 20, 21, transitivity
23. $\sim_n\forall xA(x) \vdash_{C_n^*} \sim_n\forall xA(x)$ property of $\vdash_{C_n^*}$
24. $\sim_n\forall xA(x) \vdash_{C_n^*} \exists x\sim_n A(x)$ 22, 23, MP

25. $\sim_n A(c) \vdash_{C_n*} \sim_n A(c)$	property of $\vdash_{C_n*}$
26. $\vdash_{C_n*} \sim_n A(c) \supset \sim_n A(c)$	25, Deduction Theorem
27. $\vdash_{C_n*} \exists x \sim_n A(x) \supset \sim_n A(c)$	26, Rule III[12]
28. $\sim_n \forall x A(x) \vdash_{C_n*} \sim_n A(c)$	24, 27, MP

5. Let S be of type $\sim_n(\forall x(A(x))^{(n)}) \vee (\exists x \neg(A(x)))$, generated, in ***TNDC**$_n$***, *$1\leq n<\omega$*, from $\neg\forall x A(x)$, by Rule $\neg\forall$. We have to prove that $\neg\forall x A(x) \vdash_{C_n*} \sim_n(\forall x(A(x))^{(n)}) \vee (\exists x \neg A(x))$.

1	$\vdash_{C_n*} \neg\forall x A(x) \supset (\sim_n(\forall x(A(x))^{(n)}) \vee (\exists x \neg(A(x))))$	Theorem 1.12 (viii)
2	$\neg\forall x A(x) \vdash_{C_n*} \neg\forall x A(x)$	property of $\vdash_{C_n*}$
3	$\neg\forall x A(x) \vdash_{C_n*} \sim_n(\forall x(A(x))^{(n)}) \vee (\exists x \neg A(x))$	1, 2, MP

6. Let S be of type $\sim_n(\forall x(A(x))^{(n)}) \vee (\forall x \neg A(x))$, generated, in ***TNDC**$_n$***, *$1\leq n<\omega$*, from $\neg\exists x A(x)$, by Rule $\neg\exists$. We have to prove that $\neg\exists x A(x) \vdash_{C_n*} \sim_n(\forall x(A(x))^{(n)}) \vee (\forall x \neg A(x))$.

1	$\vdash_{C_n*} \neg\exists x(A(x)) \supset (\sim_n(\forall x(A(x))^{(n)}) \vee (\forall x \neg(A(x)))$	Theorem 1.12 (ix)
2	$\neg\exists x A(x) \vdash_{C_n*} \neg\exists x A(x)$	property of $\vdash_{C_n*}$
3	$\neg\exists x A(x) \vdash_{C_n*} \sim_n(\forall x(A(x))^{(n)}) \vee (\forall x \neg A(x))$	1, 2, MP □

Hence, by Theorem 4.1 and Theorem 4.3, we have the equivalence between the correspondent systems of both hierarchies ***C**$_n$** and ***TNDC**$_n$**, $1\leq n<\omega$.

Theorem 4.4 $\Gamma \vdash_{C_n*} S$ if, and only if, $\Gamma \vdash_{TNDC_n*} S$, for every n, $1\leq n<\omega$. □

Theorem 4.5 The systems ***TNDC**$_n$***, *$1\leq n<\omega$*, constitute a hierarchy of tableaux systems, such that every system ***TNDC**$_n$*** is equivalent to da Costa's corresponding paraconsistent system ***C**$_n$***, *$1\leq n<\omega$*.

Proof. It is an immediate consequence of Theorems 4.4. □

As every system ***TNDC**$_n$***, *$1\leq n<\omega$*, is equivalent to the corresponding ***C**$_n$***, *$1\leq n<\omega$*, the syntactical and semantic results concerning the ***TNDC**$_n$*** are immediate.

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[12] Note that c is a closed term and so, it is not free in **C**.

So, the soundness and completeness of our tableaux systems can be proved.

Besides, the decidability of the monadic first-order predicate systems ***TNDC**$_n$***, $1 \leq n < \omega$, could also be proved, from the characteristics of the Expansion Rules of the systems: for every formula S we have to check, in a finite number of steps, either if $\sim_n$S is closed, or if $\sim_n$S is not closed; for every tableau for $\sim_n$S, in the case when $\sim_n$S is not closed, we have to generate at least a finite, open and complete branch. We intend to develop this proof in a future paper.

Finally, after having carefully studied da Costa's calculi $\boldsymbol{C_\omega}$ and $\boldsymbol{C_\omega}$*, we conjecture that it is not possible to present tableaux systems for such systems.

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